

ANALYTICAL MODELING AND FEM SIMULATIONS OF ENERGY HARVESTING CANTILEVER BEAM

Anastasija Ignjatovska, Damjan Pecioski, Dejan Šiškovski, Maja Anačkova,
Simona Domazetovska

Faculty of Mechanical Engineering, “Ss. Cyril and Methodius” University in Skopje,
P.O.Box 464, MK-1001 Skopje, Republic of North Macedonia
anastasija.ignjatovska@mf.edu.mk

Abstract: The concept of energy harvesting systems allows sustainable energy sources, such as mechanical energy, to be harvested from the environment. The feature of piezoelectric material to generate voltage as a result of its deformation characterizes them as an energy harvesting tool, due to their robustness and efficiency characteristics. This paper shows numerical modeling of an energy harvesting cantilever beam using the Euler-Bernoulli method. The FEM simulations of the cantilever beam have been created in order to determine the optimal position of the piezoelectric transducer along the beam. Furthermore, these conclusions have been used to develop the mathematical model in MATLAB in order to investigate the effect of the geometry characteristics of the beam and the piezoelectric transducer on the output parameters. The impact of the dimensions and effective area of the piezoelectric transducer and its location on the cantilever beam as well as the effects of the dimensions of the beam have been studied in order to obtain an optimal energy harvesting model in terms of its efficiency. Once the modeling phase has been completed, the output results regarding the generated voltage and power from the energy harvesting system have been compared and models have been validated.

Key words: energy harvesting; piezoelectric materials; cantilever beam; modeling and simulation

АНАЛИТИЧКО МОДЕЛИРАЊЕ И СИМУЛАЦИИ НА КОНЗОЛЕН СИСТЕМ ЗА СОБИРАЊЕ ЕНЕРГИЈА СО МЕТОДОТ НА КОНЕЧНИ ЕЛЕМЕНТИ

Апстракт: Концептот на системи за собирање енергија овозможува собирање на енергија од одржливи извори, каква што е механичката енергија. Пиезоелектричните материјали генерираат напон како резултат на нивната деформација, при што се карактеризираат со висока робусност и ефикасност, што овозможува нивна примена како алатка за собирање енергија. Во овој труд е извршено нумеричко моделирање на конзолан систем за собирање енергија користејќи го методот на Ојлер-Бернули. Симулациите кои се базирани на методот на конечни елементи на конзолан систем се изведени со цел определување оптимална положба на пиезоелектрикот по должината на конзолата. Понатаму овие заклучоци се искористени за развој на математички модел во MATLAB, со цел анализирање на влијанието на геометриските карактеристики на конзолата и пиезоелектрикот врз излезните параметри. Проучено е влијанието на димензиите и ефективната површина на пиезоелектрикот и неговата локација на конзолата, како и ефектите од димензиите на конзолата, со цел дизајнирање оптимален ефикасен модел за собирање енергија. Откако фазата на моделирање е завршена, добиените резултати во однос на генерираниот напон и моќност од системот за собирање енергија се споредени и моделите се валидирани.

Клучни зборови: собирање енергија; пиезоелектрични материјали; конзола; моделирање и симулации

1. INTRODUCTION

The energy crisis is one of the main problems facing humanity today as a result of which science

and researchers are constantly trying to develop sustainable solutions in order to fulfill energy demands. Therefore, technologies that use renewable energy sources such as solar energy, kinetic energy, thermal

or bio-energy noticed rapid progress. Apart from the macro-energy harvesting technologies which include renewable energy plants, recently, continuous development has also been made in microenergy harvesting technologies. Micro-energy harvesting technology is focused on the alternatives to conventional batteries and uses energy from mechanical vibration, mechanical stress and strain, which generate low-level power expressed in mW or μ W [1]. The piezoelectric energy harvesting technique has been widely researched due to its high energy conversion ability from mechanical vibration. This technique uses the properties of piezoelectric materials to generate voltage under the influence of a mechanical force.

Most research papers in the energy harvesting area, present the harvester as a cantilever beam with one or more piezoelectric layers, which is excited harmonically at its fundamental natural frequency in order to obtain maximum electrical output [2]. This is due to the fact that compared to other structures, the cantilever beam has a lower resonant frequency, and provides higher stress and strain with less ambient vibrational force [3]. There are various methods for modeling an unimorph beam, or a beam with a single piezoelectric transducer mounted on its surface. Depending on the purpose of the model, one has to determine which method is optimal. For example, Belhora *et al.* [4] use the pin-force model to develop a model that predicts the energy harvesting capabilities of an electrostrictive polymer composite. Various papers [5, 6, 7] compare the most used methods for analytical modeling: the pin-force model, the enhanced-pin force model which expands upon the pin-force concept, and the Euler-Bernoulli model. The Euler-Bernoulli method has been proven to be the most accurate representation of a real energy harvesting system out of the three methods, and for this reason it has been used in this paper. On the other hand, the finite element method has been widely used as an analysis tool for continuous systems with a finite number of concentrated masses. The discretization of the continuous systems of a finite number of elements or concentrated masses simplifies the mathematical apparatus or analytical solutions to the dynamics of the elastic structure from partial differential equations to a system of ordinary differential equations [8]. Uddin *et al.* [9] use the finite element method in determining natural frequencies, modes, and stresses along a cantilever piezoelectric energy harvesting beam. Kumar *et al.* [10] use the finite element method for coupled piezoelectric energy harvester in order to discretize the electromechanical coupling phenome-

non between mechanical and electrical domains. They optimize material properties that determine the performance of piezoelectric energy harvesters such as dielectric constant, piezoelectric strain coefficient, electromechanical coupling coefficient, Young modulus, density, and electrical and mechanical quality factors [10]. An area in energy harvesting that opens up a wide research space is the optimization of the design of the energy harvesting system in order to generate maximal voltage.

In this paper, an optimization of the cantilever piezoelectric energy harvesting beam concerning the position of the piezoelectric transducer, the position of the excitation force, and the geometrical characteristics of the piezoelectric transducer and the beam have been conducted. A combination of two different modeling methods has been used: analytical modeling using the Euler-Bernoulli method and the finite element method.

2. MATHEMATICAL MODELING

a) Mathematical modeling of the beam

Continuous systems are mechanical systems with continuously distributed mass and theoretically have infinite degrees of freedom. In order to obtain an analytical solution for the dynamics of these systems, certain approximations and simplifications have to be adopted. In order to be able to use the Euler-Bernoulli equation, the assumptions concerning the material homogeneity of the beam and its ideally constant width are adopted. Using these assumptions, partial differential equations with constant coefficients are obtained. The solutions to these partial differential equations represent the natural frequencies of the system and the modes of oscillation of the elastic structure.

The differential equation of motion of a continuous beam according to the Euler-Bernoulli method is represented with the following relation:

$$EI_y \frac{\partial^4 y(x,t)}{\partial x^4} = f(x,t) - \rho A \frac{\partial^2 y(x,t)}{\partial t^2} \quad (2.1)$$

where y is the displacement of the beam, its density is ρ_b and A_b is its area of the cross-section, whereas $\rho_b A_b \frac{\partial^2 y(x,t)}{\partial t^2}$ is kinetic energy, $E_b I_b \frac{\partial^4 y(x,t)}{\partial x^4}$ is potential energy, and $f(x,t)$ is the excitation force.

If the excitation force is a harmonic function, the differential equation can be rewritten as:

$$\frac{E_b I_b}{\rho_b A_b} \frac{\partial^4 y(x,t)}{\partial x^4} + \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{F_0}{\rho_b A_b} \sin(\omega t) \delta(x - L_f) \quad (2.2)$$

where ω is the frequency of the excitation force and L_f is the distance from the clamped end to the point where the excitation force is applied.

The general solution of the equation (2.2) can be expressed as:

$$y(x, t) = \sum_{i=1}^{\infty} T_i(t) Y_i(x) \quad (2.3)$$

where $T_i(t)$ is the i -th model coordinate of shape and $Y_i(x)$ is the i -th mode shape of the beam.

Inman *et al.* [11] state that the first 3 modes ($i = 1, 2, 3$) in equation (2.3) are sufficient for generating the mode shape equation of the cantilever beam.

By replacing expression (2.3) in (2.2), the following differential equations can be obtained [12]:

$$Y(x)^{iv} - \frac{\omega_n^2}{c^2} Y(x) = 0, \quad (2.4)$$

$$\ddot{T}(t) + \omega_n^2 T(t) = 0. \quad (2.5)$$

The solution of the first characteristic equation depends only on the x coordinate:

$$Y(x) = A \sin(\beta x) + B \cos(\beta x) + C \sinh(\beta x) + D \cosh(\beta x) \quad (2.6)$$

where β is the wave number [11] and is expressed as:

$$\beta^4 = \omega_n^2 \frac{\rho_b A_b}{E_b I_b} \quad (2.7)$$

where ω_n is the natural frequency of the system.

In the order to obtain maximal deflections of the beam, the frequency of the excitation force has to be equal to the first natural frequency of the beam. Constants A, B, C and D can be calculated using the boundary conditions for clamped-free configuration [11]. At the clamped end, for $x = 0$, slope and deflection must be zero, while at the free end, for $x = 1$, shearing force and bending moment must be equal to zero.

Finally, the general mode shape equation for a cantilever beam could be expressed as:

$$Y_i(x) = \frac{\cosh(\beta_i L_b) - \cos(\beta_i L_b) - \frac{\sinh(\beta_i L_b) - \sin(\beta_i L_b)}{\cosh(\beta_i L_b) + \cos(\beta_i L_b)} \cdot [\sinh(\beta_i L_b) - \sin(\beta_i L_b)], \quad (2.8)$$

where L_b is the total length of the beam.

Solutions for β_i (Figure 1) are calculated from the characteristic equation:

$$\cos(\beta_i L_b) \cosh(\beta_i L_b) = -1. \quad (2.9)$$

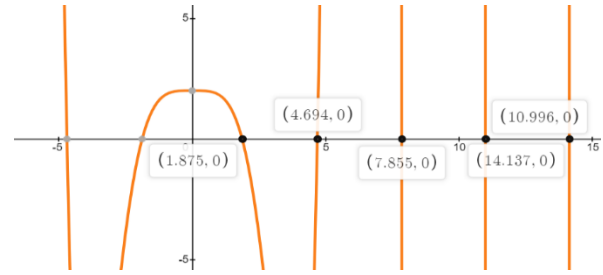


Fig. 1. Graphical representation of solutions to characteristic equation (2.9)

For the adopted value beam's length of 0.561 m. the first five natural frequencies (Table 1) can be calculated using the following expression:

$$f_i = \frac{1}{2\pi} \frac{(\beta_i L_b)^2}{L_b^2} \sqrt{\frac{E_b I_b}{\rho_b A_b}}. \quad (2.10)$$

The solution of equation (2.5) can be expressed as a convolutional integral:

$$T(t) = \frac{1}{\omega_{di}} e^{-\xi \omega_{ni} t} \int F_i(\tau) e^{-\xi \omega_{ni} \tau} \sin(\omega_{di}(t - \tau)) d\tau \quad (2.11)$$

where: ω_d is damped natural frequency, ξ is damping ratio.

Table 1

First five natural frequencies of the beam

i	$\beta_i L_b$	Natural frequencies of the beam (Hz)
1	1.875104	3.8881
2	4.696409	24.3668
3	7.854757	68.2277
4	10.995540	133.6991
5	14.137168	221.0143

The beam curvature can be calculated as:

$$\rho(x, t) = \frac{\partial^2 y(x, t)}{\partial x^2} \quad (2.12)$$

or average beam curvature:

$$\bar{\rho}(t) = \frac{1}{L_p} \int_0^{L_p} \rho(x, t) dx \quad (2.13)$$

where L_p is the total length of the piezoelectric transducer.

The applied moment acting on the beam can be expressed as:

$$M(t) = E_b I_b \bar{\rho}(t) \quad (2.14)$$

b) Mathematical modeling of PZT

A general model for the piezoelectric transducer which presents the relationship between the moment of the beam and generated voltage as a function of time is:

$$V(t) = - \frac{6g_{31} \frac{E_b t_b}{E_p t_p} \left(1 + \frac{t_b}{t_p}\right)}{b_p t_p \left\{ 1 + \left(\frac{E_b t_b}{E_p t_p}\right)^2 \cdot \left(\frac{t_b}{t_p}\right)^2 + 2 \frac{E_b t_b}{E_p t_p} \left[2 + 3 \frac{t_b}{t_p} + 2 \left(\frac{t_b}{t_p}\right)^2 \right] \right\}} \cdot M(t), \quad (2.15)$$

where $g_{31} \left(\text{V} \cdot \frac{\text{m}}{\text{N}}\right)$, is the voltage constant $E_b \left(\text{V} \cdot \frac{\text{N}}{\text{mm}^2}\right)$ is Young's modulus of the material of the beam, $I_b \left(\text{m}^4\right)$ is the axial moment of inertia of the beam and $E_p \left(\frac{\text{N}}{\text{m}^2}\right)$ is Young's modulus of the piezoelectric material. Concerning the geometry of the system, $t_b \left(\text{m}\right)$ is the beam's thickness, $t_p \left(\text{m}\right)$ and $b_p \left(\text{m}\right)$ are the thickness and width of the piezoelectric transducer, respectively.

The generated power can be calculated as:

$$P(t) = V(t) \cdot I(t) \quad (2.16)$$

$$I(t) = C_p \cdot \frac{dV}{dt} \quad (2.17)$$

$$C_p = \frac{d_{31} \cdot A_p}{g_{31} \cdot t_p} \quad (2.18)$$

where $I(t)$ is the current expressed as a function of time, C_p is the capacitance of the piezoelectric transducer, A_p is the effective area of the piezoelectric transducer, and d_{31} is its piezoelectric strain coefficient.

Based on these mathematical models of the cantilever beam and piezoelectric transducer, analytical modeling and simulations in MATLAB and FEM simulations have been performed in order to analyze this energy harvesting system in detail.

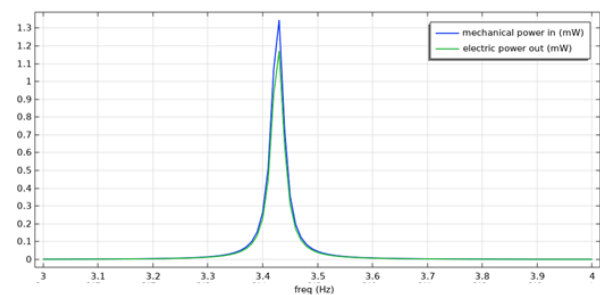
3. RESULTS AND DISCUSSION

Analytical modeling and FEM simulations have been conducted in order to analyze the impact of several input parameters on the generated output power. The following properties have been studied: position of the piezoelectric transducer along the beam, position of the excitation force, length ratio and thickness ratio of the beam and piezoelectric transducer, respectively. Obtained optimal values have been used as an input and an identical harmonic excitation force has been applied to both models. The analytical and FEM model have been validated by comparison of the obtained results concerning output voltage and power.

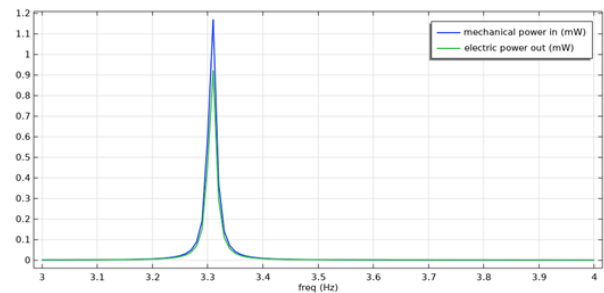
a) Position of piezoelectric transducer

Firstly, FEM simulations have been performed in order to determine the optimal position of the piezoelectric transducer along the beam, which

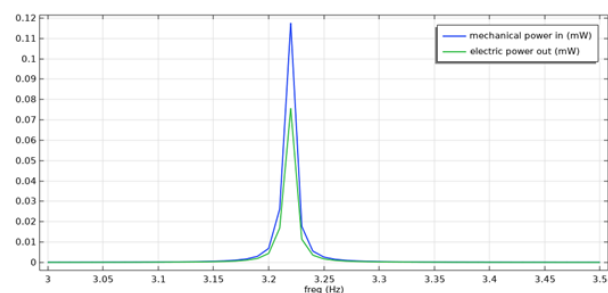
would generate maximal power. According to results from FEM simulations (Figure 2), the maximal output power is generated when the distance between the clamped end of the beam and the piezoelectric transducer equals zero. One may deduct that by increasing this distance, output electric power decreases. Therefore, optimal position of the piezoelectric transducer is next to the clamped end of the beam.



a) PZT is placed next to the clamped end



b) PZT is placed at the center of the beam



c) PZT is placed on the free end of the beam

Fig. 2. Output electric power (mW)
Three different positions of the PZT

b) Position of excitation force

Output electric power is expected to change when position of the excitation force is varied. In order to discover optimal placement of excitation harmonic force, which generates maximal output electric power, analytical modeling in MATLAB

has been performed. By increasing the distance from the clamped end of the beam to the line of action of the excitation force, the applied moment to the beam amplifies. As the obtained results from the simulation (Figure 3) show, the optimal position of the excitation force is at the free end of the beam. The excitation force can be defined with the following function:

$$F(t) = 0.35 \cdot \sin(2\pi \cdot \omega_1 \cdot t) = 0.35 \cdot \sin(2\pi \cdot 3.88 \cdot t) \quad (3.1)$$

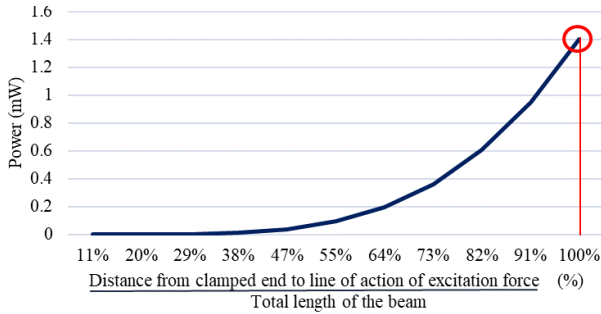


Fig. 3. Output electrical power as a function of the position of excitation force

In the optimized model the excitation force is placed at the free end of the beam and follows the function shown in (3.1).

c) Length ratio

Another parameter that has to be optimized is the length of the piezoelectric transducer in relation to the beam's length. Analytical simulations in MATLAB have been performed in order to determine optimal length of the piezoelectric transducer which enables maximal power generation. Results (Figure 4) show that when 47% of the length of the beam is covered by a PZT, maximal power is generated. Until certain value for the length of the piezoelectric transducer is reached, output power increases by increasing the length.

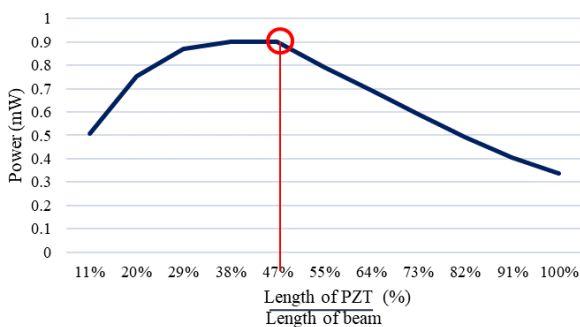


Fig. 4. Output electrical power as a function of length ratio of PZT and beam

For values of the length higher than the crossing point, overall characteristics of the system get affected and the effective cross-section, Young's modulus and natural frequencies change. Adverse effects are caused by reducing beam strain and deformation, and overall generated electric power.

d) Thickness ratio

Last parameter that has been optimized is thickness ratio of piezoelectric transducer and beam, respectively. Obtained results from MATLAB simulations (Figure 5) show a peak value of generated power for thickness ratio of 33%.

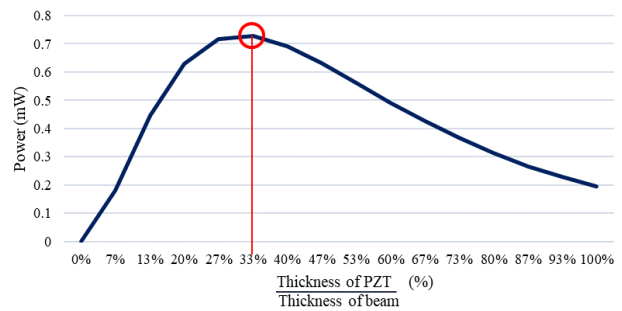


Fig. 5. Output electrical power as a function of thickness ratio of PZT and beam

For values of the thickness ratio of the piezoelectric transducer and the beam, respectively, the generated power decreases by increasing the value of thickness ratio.

e) MATLAB and FEM simulations of optimized model

Analytical modeling and simulations and FEM simulations of the optimized model (Figure 6) have been performed and output results have been compared. Identical input parameters which match calculated optimal values, concerning the dimensions and properties of the beam and the piezoelectric transducer (Table 2) have been used. The frequency of the excitation force matches the first natural frequency of the system (Figure. 6).

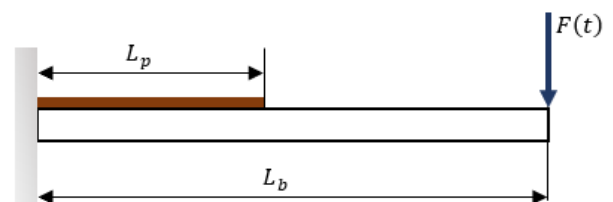


Fig. 6. Energy harvesting cantilever beam with PZT

Table 2

Parameters of the beam and PZT

	Parameter	Not.	Value	Unit
Beam	Length	L_b	0.561	m
	Width	b_b	0.04	m
	Thickness	t_b	0.0015	m
	Density	ρ_b	7850	kg/m ³
	Young's modulus	E_b	$2 \cdot 10^{11}$	Pa
PZT (PZT-5A)	Voltage constant	g_{31}	$-11.3 \cdot 10^{-3}$	V·m/N
	Dielectric constant	d_{31}	$-190 \cdot 10^{-12}$	m/V
	Young's modulus	E_p	$3 \cdot 10^3$	Pa
	Length	L_p	0.263	m
	Width	b_p	0.04	m
	Thickness	t_p	0.0005	m
	Density	ρ_p	7950	kg/m ³
	Resistance	R	1000	k Ω

Output results from the analytical modeling and simulation in MATLAB (Figure 7) show that voltage of 202.63 V and electrical power of 1.34 mW have been generated in 3 seconds, at the first natural frequency of the system. The highest values for the generated voltage and power are achieved at the first natural frequency of the system and therefore the system was excited only on that frequency.

Obtained results from the FEM simulations (Figure 8) show that in 3 seconds, voltage of 190.32 V and electrical power of 1.74 mW have been generated at the first natural frequency of the system. In the FEM simulations, until the first natural frequency has been reached, the system is excited by the excitation force given in expression (3.1). After that, the frequency of the excitation force is changed in order to excite the second natural frequency of the system (24.37 Hz). Results show that maximal output is achieved at the first natural frequency as firstly stated.

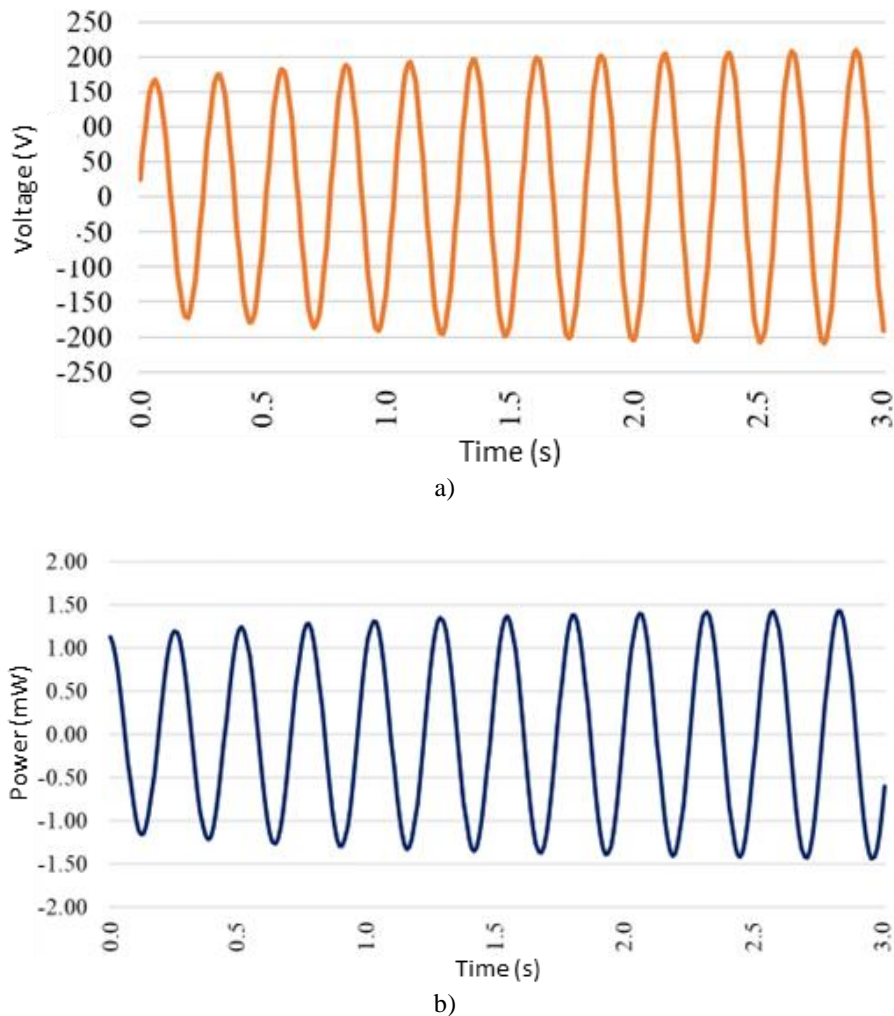


Fig. 7. Generated a) voltage (V) and b) electrical power (mW) from analytical simulations in MATLAB

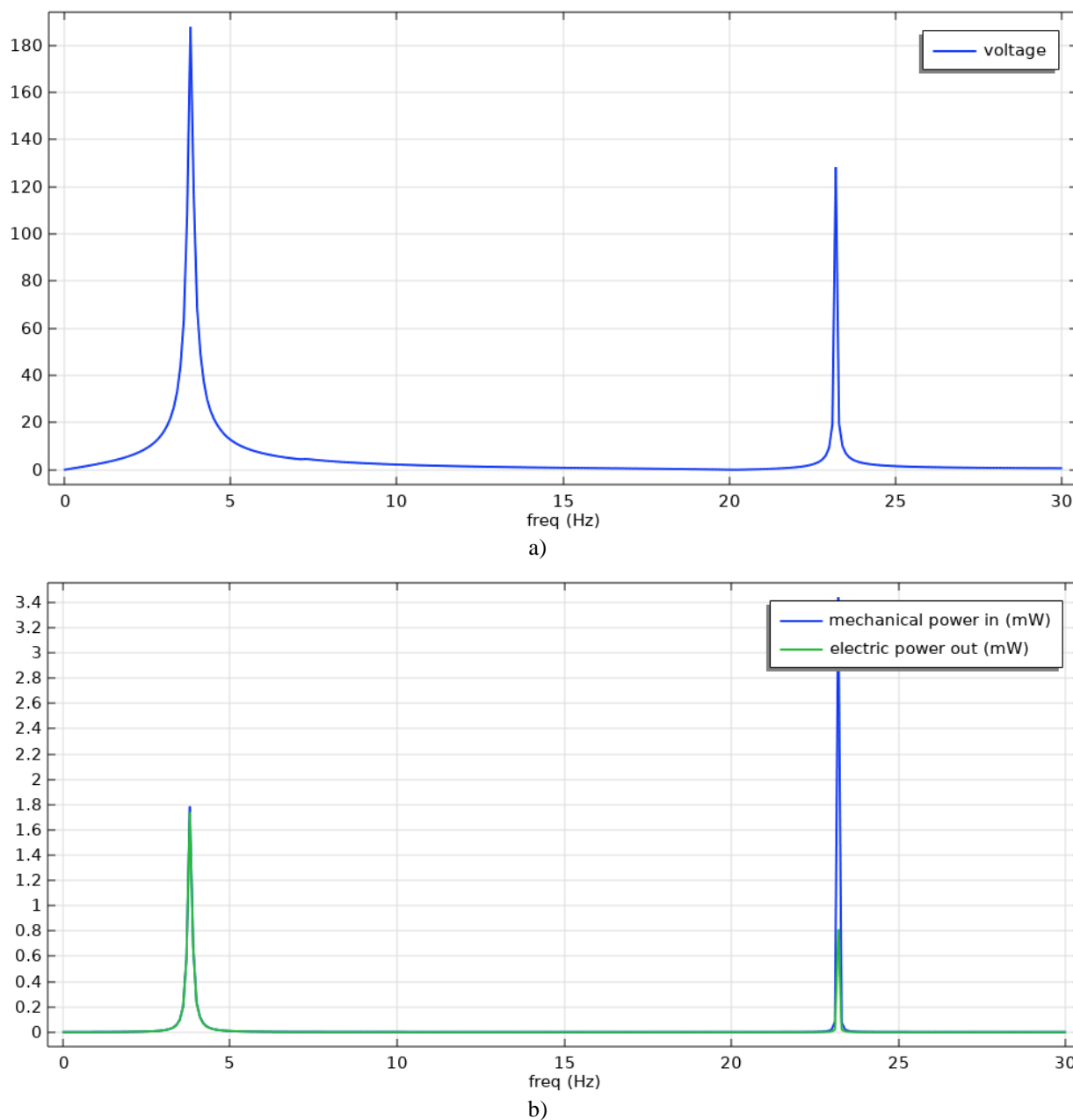


Fig. 8. Generated voltage (V) and electrical power (mW) from FEM simulations

Over a certain period of time, the generated voltage and electric power are stored and their value increases. In this paper, change of stored voltage and electric power for a period of 3 seconds has been plotted (Figure 7) and calculated (Figure 8).

Validation of the models has been performed by comparison of the obtained output results concerning the generated voltage and electrical power. Acquired results from both models using two different modeling methods have similar values and therefore model is valid. Thus, model can successfully predict power generation from a cantilever unimorph vibrating beam.

4. CONCLUSION

Analytical modeling based on Euler-Bernoulli theorem and FEM simulations of energy harvesting cantilever beam have been made. Optimal parameters of the model have been determined, concerning the position of the piezoelectric transducer, position of the excitation force, length and thickness ratio of the piezoelectric transducer and the beam, respectively. Maximal electric power has been generated for: location of the piezoelectric transducer next to the clamped end of the beam, location of excitation force on the free end of the beam, length ratio of

0.47 and thickness ratio of 0.33 between the piezoelectric transducer and the beam. Once the optimal parameters have been defined, they have been used as input parameters for the model. Based on an exact model, simulations in MATLAB and FEM simulations have been performed in order to determine output voltage and electric power. Since obtained output results for the identical model using two different techniques match, a deduction can be drawn that the model is valid.

Although, this paper is purely theoretical, the authors have also been working on an experimental setup which is based on the obtained results from the conducted software analyses. On that way, created theoretical models would be validated by an experiment. Also, an energy harvesting unimorph beam, or a beam with a single piezoelectric transducer mounted on its surface is modelled in this paper. In future work, using the proposed concept, the model could analogically be upgraded in two ways in order to generate higher voltage and electric power. First way is by creating a bimorph energy harvesting beam, by placing two PZT on the top and bottom surfaces of the beam. The second approach implies placing multiple PZT on the surface of the beam, which would enable achieving the optimal calculated length ratio of 0.47. Nevertheless, whichever approach would be chosen, proper analyses must be conducted, and adequate models must be created.

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